DUSO Mathematics League 2016 - 2017

CONTEST #5.

SOLUTIONS

5 - 1. $\boxed{\mathbf{4x}}$ The factoring is $4x^2 - 25 = (2x+5)(2x-5)$. The sum is $\mathbf{4x}$.

5 - 2. $\boxed{\dfrac{\mathbf{13}}{\mathbf{60}}}$ Consider the teens that like pop and country. To make the least fraction like all three, make the overlap as small as possible. This occurs if the fraction of teens who like pop and country is $\dfrac{4}{5} + \dfrac{3}{4} - 1 = \dfrac{11}{20}$. Now bring in the people who like jazz. Again, try to make the overlap as small as possible. This occurs if the fraction of teens who like all three kinds of music is $\dfrac{11}{20} + \dfrac{2}{3} - 1 = \dfrac{13}{60}$. The answer is $\dfrac{\mathbf{13}}{\mathbf{60}}$.

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5 - 3. $\boxed{\mathbf{147}}$ Because of the 1-1 correspondence between the triangles, the proportion that holds is $\dfrac{AC}{BC} = \dfrac{DF}{EF}$. Therefore, $\dfrac{7}{5} = \dfrac{DF}{105} \to DF = 105 \cdot \dfrac{7}{5} = 21 \cdot 7 = \mathbf{147}$.

5 - 4. $\boxed{\mathbf{10 - 4\sqrt{5}}}$ Find the equation of the circle passing through A and B tangent to the x-axis. Because the slope of line AB is $-1/3$, the equation of the line containing the center of this circle is $y - 3 = 3(x-1) \to y = 3x$, and the center has coordinates $(p, 3p)$ (notice that if P is to be a point of tangency, the center must be on a line through P perpendicular to the x-axis). The equation of the circle is $(x-p)^2 + (y-3p)^2 = (3p)^2$. Substituting the coordinates of point B yields $(4-p)^2 + (2-3p)^2 = 9p^2 \to p^2 - 8p + 16 + 9p^2 - 12p + 4 = 9p^2$, which solves to obtain $p^2 - 20p + 20 = 0$, and the only solution to this between $x = -2$ and $x = 4$ is $\mathbf{10 - 4\sqrt{5}}$.

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5 - 5. $\boxed{\mathbf{-9 - 46i}}$ First, find $(3-2i)^2 = 9 - 6i - 6i + 4i^2 = 5 - 12i$. Now, multiply $(5-12i)(3-2i) = 15 - 10i - 36i + 24i^2$, or $\mathbf{-9 - 46i}$.

5 - 6. $\boxed{\mathbf{18}}$ Write $\sum(b_k) = \sum(a_k^2 + a_k) = \sum(a_k^2) + \sum(a_k)$. The second sum is $\dfrac{3}{1 - \frac{1}{2}} = 6$. The first sum is $\sum(9 \cdot (\frac{1}{4})^{k-1}) = \dfrac{9}{1 - \frac{1}{4}} = 12$. The sum of all the terms of B is $6 + 12 = \mathbf{18}$.

R-1. Compute the smallest odd positive integer that is the product of three distinct prime numbers.
R-1Sol. $\boxed{105}$ This is found by multiplying the three smallest odd primes, so $3 \cdot 5 \cdot 7 = \mathbf{105}$.

R-2. *Let N be the number you will receive.* The quadratic equation $x^2 - 8x - N = 0$ has two roots. Compute the greater of these two roots.
R-2Sol. $\boxed{15}$ The equation is $x^2 - 8x - 105 = 0 \rightarrow (x-15)(x+7) = 0$, so the greater of the two roots is **15**.

R-3. *Let N be the number you will receive.* An arithmetic sequence begins $18, N, \cdots$. The difference between any two consecutive terms is constant. Compute the fifth term in the sequence.
R-3Sol. $\boxed{6}$ The common difference is $N - 18$, and the fifth term is $18 + 4(N-18) = 4N - 54$. Substituting and solving, the fifth term is $4 \cdot 15 - 54 = \mathbf{6}$.

R-4. *Let N be the number you will receive.* A cylindrical can with a closed top and closed bottom has a surface area of $N\pi$ square cm and a base radius of 1 cm. Compute the volume of the can in cubic cm.
R-4Sol. $\boxed{2\pi}$ The surface area is $2\pi \cdot 1^2 + 2\pi \cdot 1 \cdot h = N\pi$, so $h = \dfrac{(N-2)\pi}{2\pi} = \dfrac{N-2}{2}$. The volume is thus $V = \pi 1^2 \cdot \dfrac{N-2}{2}$. Substituting yields $V = \mathbf{2}\pi$.

R-5. *Let N be the number you will receive.* The lateral area of a right circular cone is $N/2$. The height of the cone is $\dfrac{\sqrt{15}}{2}$. Compute the radius of the cone.
R-5Sol. $\boxed{\frac{1}{2}}$ The lateral area is $\pi r l = N/2$, so $\pi r \sqrt{r^2 + 15/4} = N/2$. This implies that $4\pi^2 r^2 (r^2 + 15/4) = N^2$, and by substitution, $r^2(r^2 + 15/4) = 1$. This has two solutions, but one is negative, so $r = \dfrac{\mathbf{1}}{\mathbf{2}}$.

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